

Chevalley restriction theorem for the cyclic quiver

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Abstract

We prove a Chevalley restriction theorem and its double analogue for the cyclic quiver.

The aim of this paper is to prove a Chevalley restriction theorem and its double analogue for the cyclic quiver. When the quiver is of type \hat{A}_0 , we recover the results for \mathfrak{gl}_n . The proof of our Chevalley restriction theorem is similar to the proof for \mathfrak{gl}_n ; however, the proof of the double analogue uses a theorem of Crawley-Boevey on decomposition of quiver varieties. The double analogue is the limiting case of an isomorphism between a Calogero-Moser space and the center of a symplectic reflection algebra proved by Etingof and Ginzburg. It is also the associated graded version of a conjectural Harish-Chandra isomorphism for the cyclic quiver.

We now introduce our notations. Let Q be the cyclic quiver with m vertices. Let $\delta = (1, \dots, 1)$ be the minimal positive imaginary root. Let $\mathcal{R}_n = \text{Rep}(Q, n\delta)$ be the space of representations of Q with dimension vector $n\delta$. Thus,

$$\mathcal{R}_n = \underbrace{\mathfrak{gl}_n \times \cdots \times \mathfrak{gl}_n}_m.$$

Next, let \mathfrak{h} be the subspace of diagonal matrices in \mathfrak{gl}_n , and let

$$\mathcal{L}_n = \{(z, \dots, z) \in \mathcal{R}_n \mid z \in \mathfrak{h}\}.$$

Note that \mathcal{L}_n is a n dimensional subspace of \mathcal{R}_n . Let

$$G_n = \underbrace{GL_n \times \cdots \times GL_n}_m.$$

An element $(g_1, \dots, g_m) \in G_n$ acts on an element $(x_1, \dots, x_m) \in \mathcal{R}_n$, giving

$$(g_2^{-1}x_1g_1, g_3^{-1}x_2g_2, \dots, g_1^{-1}x_mg_m).$$

Let \mathbb{S}_n be the symmetric group on n letters, which we will also regard as the subgroup of permutation matrices in GL_n . Finally, let

$$W_n = \mathbb{S}_n \ltimes (\mathbb{Z}/m\mathbb{Z})^n.$$

We have $W_n \hookrightarrow G_n$ via

$$(\sigma, \zeta_1, \dots, \zeta_n) \mapsto (\sigma \cdot \text{diag}(1, \dots, 1), \sigma \cdot \text{diag}(\zeta_1, \dots, \zeta_n), \dots, \sigma \cdot \text{diag}(\zeta_1^{m-1}, \dots, \zeta_n^{m-1})),$$

where $\text{diag}(\dots)$ denotes the diagonal matrix with the indicated entries. Hence, W_n acts on \mathcal{R}_n . Observe that the action of W_n on \mathcal{L}_n is stable. We remark that W_n is the complex reflection group of type $G(m, 1, n)$ and \mathcal{L}_n is its reflection representation.

Theorem 1 (Chevalley restriction). *Restriction of functions from \mathcal{R}_n to \mathcal{L}_n gives an isomorphism*

$$\rho : \mathbb{C}[\mathcal{R}_n]^{G_n} \xrightarrow{\sim} \mathbb{C}[\mathcal{L}_n]^{W_n}.$$

Proof. Surjectivity of ρ : Write an element in \mathcal{R}_n as (x_1, \dots, x_m) and an element in \mathcal{L}_n as

$$(\text{diag}(z_1, \dots, z_n), \dots).$$

Note that $\mathbb{C}[\mathcal{L}_n]^{W_n}$ is a polynomial algebra generated by the elementary symmetric polynomials in z_1^m, \dots, z_n^m . The homomorphism ρ takes the coefficients of the characteristic polynomial of $x_m x_{m-1} \cdots x_1$ to the elementary symmetric polynomials in z_1^m, \dots, z_n^m . This proved that ρ is surjective.

Injectivity in the $n = 1$ case: Call an element $(x_1, \dots, x_m) \in \mathcal{R}_1$ generic if $x_1 \cdots x_m \neq 0$. Observe that the set of generic elements are Zariski open dense in \mathcal{R}_1 . Moreover, it is easy to see that in this case, two generic elements (x_1, \dots, x_m) and (x'_1, \dots, x'_m) are in the same G_1 -orbit iff $x_1 \cdots x_m = x'_1 \cdots x'_m$. In particular, \mathcal{L}_1 intersects every generic orbit. Hence, ρ is injective.

Injectivity in the general case: Call an element (x_1, \dots, x_m) in \mathcal{R}_n generic if $x_m x_{m-1} \cdots x_1$ has pairwise distinct nonzero eigenvalues. Denote the subset of generic elements in \mathcal{R}_n by \mathcal{R}'_n , and let $\mathcal{L}'_n = \mathcal{L}_n \cap \mathcal{R}'_n$. Observe that \mathcal{R}'_n and \mathcal{L}'_n are, respectively, Zariski open dense in \mathcal{R}_n and \mathcal{L}_n . Moreover, \mathcal{R}'_n is G_n -stable and \mathcal{L}'_n is W_n -stable. The injectivity of ρ follows from the $n = 1$ case and the following claim.

Claim: If $(x_1, \dots, x_m) \in \mathcal{R}'_n$, then it can be diagonalized, i.e. G_n -conjugated to an element in

$$\underbrace{R_1 \times \cdots \times R_1}_n = \underbrace{\mathfrak{h} \times \cdots \times \mathfrak{h}}_m.$$

Proof of Claim: By our assumption, x_1, \dots, x_m are invertible matrices. Moreover, there exists an invertible matrix g such that $g^{-1} x_m x_{m-1} \cdots x_1 g$ is diagonal. Then, using

$$(g, x_1 g, x_2 x_1 g, \dots, x_{m-1} \cdots x_1 g) \in G_n,$$

we can conjugate (x_1, \dots, x_m) to

$$(1, \dots, 1, g^{-1}x_mx_{m-1} \cdots x_1g).$$

This proved the claim, and hence the theorem. \square

Remark 2. The Jacobian of the morphism $\mathcal{L}_n \rightarrow \mathcal{L}_n/W_n$ at a point

$$(z, \dots) = (\text{diag}(z_1, \dots, z_n), \dots) \in \mathcal{L}_n$$

is, up to a nonzero constant, equal to

$$(z_1 \cdots z_n)^{m-1} \prod_{i < j} (z_i^m - z_j^m).$$

Thus, \mathcal{L}'_n is the set of points where the Jacobian is nonzero.

We now proceed to the double analogue of Theorem 1. Let \mathcal{Z}_n be the zero set of the moment map of the G_n -action on $T^*\mathcal{R}_n = \text{Rep}(\overline{Q}, n\delta)$, where \overline{Q} is the double quiver of Q . Write an element in $\text{Rep}(\overline{Q}, n\delta)$ as

$$(x_1, \dots, x_m, y_1, \dots, y_m) \in \underbrace{\mathfrak{gl}_n \times \cdots \times \mathfrak{gl}_n}_{2m}.$$

Here, the arrow for y_i is opposite to the arrow for x_i . In explicit terms, \mathcal{Z}_n is defined by the moment map equations

$$y_1x_1 - x_my_m = 0, \ y_2x_2 - x_1y_1 = 0, \ \dots$$

The action of an element $(g_1, \dots, g_m) \in G_n$ on $\text{Rep}(\overline{Q}, n\delta)$ is given by the formula

$$(g_2^{-1}x_1g_1, \dots, g_1^{-1}x_mg_m, \ g_1^{-1}y_1g_2, \dots, \ g_m^{-1}y_mg_1).$$

Note that \mathcal{Z}_n is stable under the G_n -action, and $\mathcal{L}_n \times \mathcal{L}_n \subset \mathcal{Z}_n$.

Theorem 3 (Double analogue). *Restriction of functions from \mathcal{Z}_n to $\mathcal{L}_n \times \mathcal{L}_n$ gives an isomorphism*

$$\phi : \mathbb{C}[\mathcal{Z}_n]^{G_n} \xrightarrow{\sim} \mathbb{C}[\mathcal{L}_n \times \mathcal{L}_n]^{W_n}.$$

Proof. Surjectivity of ϕ : Write an element of $\mathcal{L}_n \times \mathcal{L}_n$ as

$$(\text{diag}(z_1, \dots, z_n), \dots, \text{diag}(z'_1, \dots, z'_n), \dots).$$

By a result of Weyl [We], the algebra $\mathbb{C}[\mathcal{L}_n \times \mathcal{L}_n]^{W_n}$ is generated by

$$z_1^r z_1'^s + \cdots + z_n^r z_n'^s,$$

where $r, s \geq 0$ and $r - s$ is divisible by m . The homomorphism ϕ takes

$$\text{Tr}(\underbrace{y_1 \cdots y_m y_1 \cdots y_m \cdots y_1 \cdots y_j}_{s} \underbrace{x_j \cdots x_1 \cdots x_m \cdots x_1 x_m \cdots x_1}_{r})$$

to $z_1^r z_1'^s + \cdots + z_n^r z_n'^s$. Hence, ϕ is surjective.

Injectivity in the $n = 1$ case: Suppose $x_1 \cdots x_m = d^m \neq 0$ for some d . Let $g_i = x_1 \cdots x_{i-1}/d^{i-1}$. Then $g_{i+1}^{-1} x_i g_i = d$ and $g_i^{-1} y_i g_{i+1} = x_i y_i / d$. But $x_1 y_1 = x_2 y_2 = \cdots = x_m y_m$ by the moment map equations. Hence, the G_1 -saturation of $\mathcal{L}_1 \times \mathcal{L}_1$ is Zariski dense in \mathcal{Z}_1 . It follows that ϕ is injective.

Injectivity in the general case: Observe that

$$\underbrace{\mathcal{Z}_1 \times \cdots \times \mathcal{Z}_1}_n = (\underbrace{\mathfrak{h} \times \cdots \times \mathfrak{h}}_m) \cap \mathcal{Z}_n \hookrightarrow \mathcal{Z}_n.$$

This inclusion induces the map f in the following commutative diagram.

$$\begin{array}{ccc} \mathbb{C}[\mathcal{Z}_n]^{G_n} & \xrightarrow{f} & ((\mathbb{C}[\mathcal{Z}_1]^{G_1})^{\otimes n})^{\mathbb{S}_n} \\ \phi \downarrow & & \downarrow \psi \\ \mathbb{C}[\mathcal{L}_n \times \mathcal{L}_n]^{W_n} & \longrightarrow & ((\mathbb{C}[\mathcal{L}_1 \times \mathcal{L}_1]^{W_1})^{\otimes n})^{\mathbb{S}_n} \end{array}$$

In this diagram, the map f is injective by [CB, Theorem 3.4], and the map ψ is injective by the $n = 1$ case which we have proved. Hence, ϕ must be injective. \square

When Q is of type \widehat{A}_0 , i.e. the \mathfrak{gl}_n -case, the injectivity of ϕ was due to Gerstenhaber [Ge], cf. [Ri].

Remark 4. Let Γ be any finite subgroup of $SL_2(\mathbb{C})$ and let $\mathbf{\Gamma}_n = \mathbb{S}_n \ltimes \Gamma^n$. In [EG], Etingof and Ginzburg defined the Calogero-Moser space $\mathcal{M}_{\Gamma, n, c}$ associated to Γ and a parameter c . They proved that, for generic c , there is an isomorphism $\mathbb{C}[\mathcal{M}_{\Gamma, n, c}] \xrightarrow{\sim} \mathbb{Z}_{0, c}(\mathbf{\Gamma}_n)$, where $\mathbb{Z}_{0, c}(\mathbf{\Gamma}_n)$ is the center of the symplectic reflection algebra associated to the group $\mathbf{\Gamma}_n$, see [EG, Theorem 11.16]. The isomorphism ϕ in Theorem 3 is the $c = 0$ case of their isomorphism when $\Gamma = \mathbb{Z}/m\mathbb{Z}$; it is also the associated graded version of the isomorphism in [EG, Conjecture 11.22] for this Γ .

Acknowledgment. I thank Victor Ginzburg for suggesting the problem and for helpful discussions.

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